MAT237 Midterm 1

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Definitions/Examples

For each part, give the definition of the indicated term, and provide an example as indicated. In everything that follows, let $S \subseteq \mathbb{R}^n$. Each question is worth 2 points, one point for the definition, and one point for examples/counterexamples.

(1) Define what it means for S to be *open*. Give an example of an open set. Give an example of a set which is both open and closed.

(2) Define the *boundary* of S. Give an example of a set S such that there is a unique point x such that $x \in \partial S$, but $x \notin S$.

(3) Define what it means for S to be *disconnected*. Give one example of a connected set, and one example of a disconnected set.

(4) State the definition of what it means for $f : S \to \mathbb{R}$ to be *uniformly continuous*. Give an example of a function which is continuous, but not uniformly continuous.

(5) Give the definition of a *Cauchy sequence*. Give an example of a sequence which is not Cauchy.

(6) Define what it means for S to be compact. Give an example of a compact set.

True of False

For each part, state whether or not the given sentence is true or false. If true, give a short proof why. If false, provide a counterexample or an explanation of why the statement is false. (2 points each, one point for T/F, one point for explanation/example).

(1) Every bounded below subset of $\mathbb{Q}$ has a greatest lower bound in $\mathbb{Q}$.

(2) The open unit ball $B(0,1) = \{x \in \mathbb{R}^n \mid |x| < 1\}$ is a convex set.

(3) If $S = \big\{(x,y) \in \mathbb{R}^2 \mid x > 0\big\}$, then S has empty interior.

(4) Define:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Then $f(x,y)$ is continuous at $(0,0)$.

Let $S = \{(x, y) \in \mathbb{R}^2 \mid y = \sin(1/x)\}$.

(1) (2 points) Sketch S. Use solid lines to indicate points which are in S, and dotted lines to indicate points on the boundary which are not in S.

(2) (4 points) What is the boundary of S? Provide proof.

(3) (4 points) Is S open, closed, neither open nor closed, or both open and closed? Explain your choice.

(10 points) Determine whether or not each of the following sequences converges or diverges. If a sequence converges, find its limit. If a sequence does not converge, provide a proof.

(1)

$$(x_k, y_k) = (\cos \pi k, \sin \pi k) \subseteq \mathbb{R}^2$$

(2)

$$(x_k, y_k) = \left(\frac{\cos \pi k}{k}, \frac{\sin \pi k}{k} \right) \subseteq \mathbb{R}^2$$

(3)

$$x_k = \frac{(-1)^k}{k^2} \subseteq \mathbb{R}$$

(4)

$$(x_k, y_k, z_k) = \left(\frac{1}{k}, \frac{\sin k}{k}, k \right) \subseteq \mathbb{R}^3$$

(1) (6 points) Show that if U and V are open sets in $\mathbb{R}^n$, then both $U \cup V$ and $U \cap V$ are open.

(2) (4 points) If $\{U_i\}_{i=1}^{\infty}$ is an infinite collection of open sets in $\mathbb{R}^n$, then must $\bigcup_{i=1}^{\infty} U_i$ be open? Must $\bigcap_{i=1}^{\infty} U_i$ be open? Provide a proof or a counter-example in each case.

(10 points) If $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence and $f : \mathbb{R}^n \to \mathbb{R}$ is uniformly continuous, then $\{f(x_k)\}_{k=1}^{\infty}$ is Cauchy.

(1) (3 points) State the Bolzano-Weierstrass theorem

(2) (7 points) Show that if $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous and $K \subseteq \mathbb{R}^n$ is compact, then $f(K)$ is compact.

(7 points) Let $C([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous on } [0,1]\}$. There is a notion of "distance" between two continuous functions $f, g \in C([0,1])$:

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)| \quad \forall\, f, g \in C([0,1])$$

Fix a function $g \in C([0,1])$, then let $U = \{f \in C([0,1]) \mid f(x) < g(x)\, \forall\, x \in [0,1]\}$. Show that U is open in $C([0,1])$ with respect to the given distance function. (HINT: Think about the definition of an open set in $\mathbb{R}^n$ and how you would show that a given set is open. Imitate that proof in this context)

(3 points) Is U still open if $C([0,1])$ had been $C(\mathbb{R})$, continuous functions on the reals?

(BONUS 3 points: Show that U is connected)